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Memoir on a New Theory of Symmetric Functions.

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In a communication recently made to the London Mathematical Society, I have sketched out an extension of the algebra of the theory of symmetrical functions and have established the bases of a wide development.

For the sake of unity, as well as for the convenience of the readers of this journal, I repeat some definitions and preliminary theorems which are of great moment to the due comprehension of what follows.

The main object of this memoir is to show clearly the proper place of the "Symmetric Function Tables" as studied by Hirsch, Cayley, Durfee and other mathematicians in Europe and America, in the algebra of such functions; to point out that the fact of their existence depends upon a wide theorem of algebraic reciprocity which leads to an equally wide theorem of algebraic expressibility, and that they are a particular case, and not the most important case from the point of view of application, of a system of such tables.

I indicate an application to the general theory of binary forms, which as regards ground forms and syzygies is of considerable promise.

It has been usual to discuss and develop the theory by reference to the weight of the involved symmetric functions; tables have thus been constructed of weights 1, 2, 3, ... , and laws and formulae appertaining thereto have been evolved; some of these laws and formulae are in regard to an arbitrary weight, but so far as my knowledge extends no attempt has hitherto been made to develop the theory from a more general standpoint.

In what follows, I regard the whole theory as arising from the discussion of an arbitrary partition of an arbitrary weight, and bring out the ordinary theory as the particular case corresponding to that partition of w (the arbitrary weight) which is composed of w units.

The relation of the ordinary to the present inclusive theory will become clear as the investigation proceeds, and it will in particular be interesting to note those theorems which are perfectly general for every partition, and those which are peculiar to single partitions.

Definitions.

A number is partitioned into parts by writing down a set of positive numbers which result in the number when added together; each of the constituent numbers is a part of the partition, and the parts are usually written in descending order from left to right and enclosed in a bracket ( ).

A partition is separated into separates by writing down a set of partitions, each separate partition in its own brackets, from left to right, so that when all the parts of these partitions are assembled in a single bracket, the partition which is separated is reproduced. Thus of a partition

$$(p_1p_2p_3p_4p_5),$$

separations are

$$(p_1p_2)(p_3p_4)(p_5),$$
$$(p_1p_2p_3)(p_4p_5).$$

Professor Sylvester has termed a number, quâ its partitions, the partible number; so here we may term a partition, quâ its separations, the separable partition. It is convenient to order the separates of a separation from left to right, in descending order as regards weight.

If the successive weights of the separates be

$$w_1,\ w_2,\ w_3,\ \ldots\ldots,$$

I speak of a separation of species partition

$$(w_1w_2w_3\ \ldots\ldots).$$

The sum of the highest parts of the several separates I further call the "degree" of the separation. The characteristics of a separation are

(i) the weight,
(ii) the separable partition,
(iii) the species partition,
(iv) the degree;

to which may be added

(v) the multiplicity,

where, if the separation be

$$(p_1 \ldots .)^{j_1}(p_s \ldots .)^{j_2}(p_t \ldots .)^{j_3} \ldots .,$$

the multiplicity is defined by the succession of indices

$$j_1 j_2 j_3 \ldots .$$

Theorem of Reciprocity.

Partitions being symbolical representations of symmetric functions, let

$$X_1 = (1)\, x_1,$$
$$X_2 = (2)\, x_2 + (1^2)\, x_1^2,$$
$$X_3 = (3)\, x_3 + (21)\, x_2 x_1 + (1^3)\, x_1^3,$$
$$X_4 = (4)\, x_4 + (31)\, x_3 x_1 + (2^2)\, x_2^2 + (21^2)\, x_2 x_1^2 + (1^4)\, x_1^4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$X_m = \sum (m_1 m_2 m_3 \ldots .)\, x_{m_1} x_{m_2} x_{m_3} \ldots .,$$

the summation being in regard to every partition

$$(m_1 m_2 m_3 \ldots .)$$

of the number m.

Form the product $$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots .$$

and observe that, on performing the multiplication, the coefficient of the x term

$$x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots .$$

is a sum of compound symmetric functions, each of which, to a numerical factor près, is represented symbolically by a separation of the partition

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots .),$$

and further, that each such separation has a species partition

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots .).$$

Write then $$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots . = \sum P x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \ldots . ;$$

P, being a sum of compound symmetric functions, can be expanded in a sum of monomial symmetric functions. Thus, suppose

$$P = \sum \theta\, (\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots .),$$

we may write

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \ldots . = \sum \sum \theta\, (\lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} \ldots .)\, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_3} x_{s_3}^{\sigma_3} \ldots .$$

I have established, in a practically instantaneous manner (loc. cit.), by considering a certain restricted distribution of objects amongst parcels, that in this result the coefficient θ remains unaltered if we interchange the species partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots\dots)$$

and the partition $(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\dots\dots)$;
so that we have also

$$X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}X_{\lambda_3}^{l_3}\dots\dots=\sum\sum\theta\,(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots\dots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\dots\dots;$$

in other words, the theorem of reciprocity states that the coefficient of

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\dots\dots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\dots\dots$$

in the development of $X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\dots\dots$
is equal to the coefficient of

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots\dots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\dots\dots$$

in the development of $X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}X_{\lambda_3}^{l_3}\dots\dots$

Observe that the Cayley-Betti law of symmetry connected with ordinary tables of symmetric functions is obtained from this theorem by merely attending to the powers of x_1.

Formation of New Tables.

Writing as before

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\dots\dots=\sum Px_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\dots\dots,$$

$$P=\sum\theta\,(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3})\dots\dots,$$

P is an aggregate of separations of

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\dots\dots)$$

of species partition $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots\dots)$.
Of any separable partition $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\dots\dots)$

there is a definite number of species partitions, which is in general less than the number of separations. Forming a product

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\dots\dots$$

for each such species partition, we get an equal number of expressions P, each of which is an aggregate of separations of the corresponding species; on expanding all these expressions P in a series of monomials, we obtain a certain number of different symmetric functions

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\dots\dots),$$

each of which, by the law of reciprocity, is a species partition of the separable partition $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots.)$;
hence if there be altogether k species partitions of the separable partition

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots.),$$

the development of the corresponding k products

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots.$$

will, through the k expressions P, lead to precisely k different monomial symmetric functions $(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots.)$
symbolized by identically the same partitions as the species partitions; hence we see directly that, given a separable partition

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots.),$$

the species partitions $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots.)$
are, in some order, the same as the partitions

$$(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots.),$$

and that writing the expressions

$$P$$

in any order vertically and the corresponding species partitions in the same order from left to right, we are able to express the separations

$$P$$

in terms of monomial symmetric functions by means of a table possessing the same row and column symmetry as given by the Cayley-Betti law in the particular case of existing tables.

Theorem: Of a separable partition

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots.)$$

the separations P

are expressible by means of monomial symmetric functions, symbolized by partitions which are identical with the species partitions, and a symmetrical table may be thus formed.

By solving a set of simultaneous linear equations, we may now express the monomial symmetric functions which are symbolized by the species partitions of the separations of $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\dots)$ in terms of the separations P which arise from the X-products.

We must also thus get a table possessing the same law of symmetry, as may be easily gathered from the elementary properties of determinants.

Theorem: The monomial symmetric functions, symbolized by the species partitions of the separations of

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\dots)$$

are expressible in terms of compound symmetric functions, symbolized by the separations P which arise from the X-products corresponding to the several species partitions, and a table thus formed will possess row and column symmetry.

This last theorem is in fact a law of algebraic expressibility which may be further enunciated as follows:

Theorem of expressibility: If a symmetric function be symbolized by

$$(\lambda\mu\nu\dots)$$

and $(\lambda_1\lambda_2\lambda_3\dots)$ be any partition of λ,
$(\mu_1\mu_2\mu_3\dots)$ " " " μ,
$(\nu_1\nu_2\nu_3\dots)$ " " " ν,
. . . .

the symmetric function $(\lambda\mu\nu\dots)$ is expressible by means of separations of

$$(\lambda_1\lambda_2\lambda_3\dots\mu_1\mu_2\mu_3\dots\nu_1\nu_2\nu_3\dots).$$

In the ordinary theory, we have the particular case that every symmetric function whatever is expressible by means of the elementary symmetric functions (that is, by the coefficients of the equation, the symmetric functions of whose roots we are considering); observe that these coefficients have partitions composed wholly of units, and that the theorem necessarily arises because *every* member is expressible as a sum of units; the theorem in this case is also one of reducibility, but in general expressibility is not coincident with reducibility, as will appear later on.

I now proceed to show the practical method of constructing symmetrical symmetric function tables corresponding to any partition of any number. For clearness I will take a simple case, viz the partition

$$21^3$$

of the number 5.

We are concerned with symmetric functions of weight 5. Write down the separations of 21^3, viz:

(21^3) of species partition (5),
$(21^2)(1)$ " " (41),
$\left.\begin{matrix}(21)(1^2)\\(1^3)(2)\end{matrix}\right\}$ " " (32),
$(21)(1)^2$ " " (31^2),
$(2)(1^2)(1)$ " " (2^21),
$(2)(1)^3$ " " (21^3).

To each of these separations must be attached the numerical coefficient which arises from the X-product corresponding to the species partition. Thus to find the coefficient of the separation

$$(2)(1^2)(1)$$

we form the product

$$X_2^2X_1 \equiv \{(2)x_2 + (1^2)x_1^2\}^2\{(1)x_1\},$$

because (2^21) is the species partition of

$$(2)(1^2)(1),$$

and pick out the coefficient of

$$(2)(1^2)(1)\,x_2x_1^3$$

in its development. Since

$$X_2^2X_1 = \ldots\ldots + 2\,(2)(1^2)(1)\,x_2x_1^3 + \ldots\ldots,$$

the required coefficient is 2.

In this manner the proper coefficient of each separation has to be determined, and they may then be written down in a vertical column according to the dictionary, alphabetical or Durfee-order of their species partitions as may seem appropriate to the purpose in hand.

The species partitions themselves are then written in the *same order* from left to right and the skeleton table then appears in the form :

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)
(21^3)						
$(21^2)(1)$						
$(21)(1^2)+(1^3)(2)$						
$(21)(1)^2$						
$2(2)(1^2)(1)$						
$(2)(1)^3$						

a table which, when filled in, will express the separations in terms of the monomial symmetric functions in a symmetrical manner.

The companion table will express inversely the monomial symmetric functions in terms of the separations, and will have the skeleton form :

	(21^3)	$(21^2)(1)$	$(21)(1^2)+(1^3)(2)$	$(21)(1)^2$	$2(2)(1^2)(1)$	$(2)(1)^3$
(5)						
(41)						
(32)						
(31^2)						
(2^21)						
(21^3)						

This example will, I think, make the method in general clear; we can readily obtain the coefficient to be used with any particular separation by considering the corresponding X-product. Thus, suppose the separation to be

$$(a_1)^{\alpha_1}(a_2)^{\alpha_2}\dots\dots(b_1)^{\beta_1}(b_2)^{\beta_2}\dots\dots(c_1)^{\gamma_1}(c_2)^{\gamma_2}\dots\dots,$$

wherein $a_1, a_2, \dots\dots$ are partitions each of weight a,
$b_1, b_2, \dots\dots$ " " " b,
$c_1, c_2, \dots\dots$ " " " c,
. . . .

then we have to put the numerical coefficient of

$$(a_1)^{\alpha_1}(a_2)^{\alpha_2}\ldots\ldots(b_1)^{\beta_1}(b_2)^{\beta_2}\ldots\ldots(c_1)^{\gamma_1}(c_2)^{\gamma_2}\ldots\ldots$$

in the development of the product

$$X_a^{\alpha_1+\alpha_2+\ldots\ldots}X_b^{\beta_1+\beta_2+\ldots\ldots}X_c^{\gamma_1+\gamma_2+\ldots\ldots}\ldots\ldots,$$

which is $\dfrac{(\alpha_1+\alpha_2+\ldots\ldots)!\,(\beta_1+\beta_2+\ldots\ldots)!\,(\gamma_1+\gamma_2+\ldots\ldots)!\ldots\ldots}{\alpha_1!\,\alpha_2!\ldots\ldots\quad\beta_1!\,\beta_2!\ldots\ldots\quad\gamma_1!\,\gamma_2!\ldots\ldots}$.

The process just laid down for the formation of symmetrical tables is that which would be adopted *a priori* as a result of the general law of reciprocity; there is, however, a practical convenience in modifying the process, in a manner which in no way interferes with the symmetrical character of the tables, so as either to lessen the magnitude of the table numbers or to get rid of fractions.

In the tables which express the separations in terms of the monomials, no fractions can possibly occur, but the numbers may be reduced in magnitude by a modification now to be explained.

In the table which expresses the separations of (21^3), the skeleton of which is given above, the symmetry will remain unchanged if we simultaneously write

$$(2)(1^2)(1) \text{ for } 2(2)(1^2)(1)$$

in the vertical column, and

$$2(2^21) \text{ for } (2^21)$$

in the horizontal row. The effect of this is to diminish the numbers without introducing fractions. To see the reason of this, observe that the coefficient of

$$(2^21)$$

in the development of every separation is necessarily of the form

$$0 \bmod 2$$

(vide Cayley, *Amer. Jour. of Math.*, Vol. VII, p. 59), so that in the original form of table, the vertical column headed (2^21) is necessarily divisible by 2; further, the horizontal row $2(2)(1^2)(1)$ is obviously simultaneously divisible by 2 also.

These divisions will be accomplished, as stated above, by simultaneously writing $(2)(1^2)(1)$ for $2(2)(1^2)(1)$ and $2(2^21)$ for (2^21).

Generally, if in the vertical column any separation occur singly with a coefficient p, the coefficient of the corresponding species partition in the development of every separation of the separable partition is of the form

$$0 \bmod p$$

(vide Cayley loc. cit.); we can thus divide the separation row and the corresponding species partition column simultaneously by p, and this clearly does not interfere with the symmetry. Suppose now that we have in the vertical column of separations an aggregate of separations of a certain species partition, each separation having attached to it its proper coefficient as obtained from the X-product; of these separations, a certain one may be termed the leading separation. To make clear what I mean, I must explain what I wish to be understood by the "leading term" in the development of a separation. Taking any separation, say

$$(51)(432)(1^5),$$

write down the separates in a column, thus

$$\begin{array}{r} 51 \\ 432 \\ 11111 \\ \hline 10,5311 \end{array}$$

and add up the parts vertically, thus obtaining the monomial

$$(10,531^2).$$

I call this term $(10,531^2)$ the leading term in the development of the separation

$$(51)(432)(1^5),$$

and observe particularly that this term, in the development, of necessity occurs with a coefficient unity and precedes in dictionary order any other term arising in the development.

On this understanding I call the leading separation of an aggregate of separations of the same species partitions, that separation whose leading term precedes in dictionary order the leading terms of all other separations in the aggregate. (As to "dictionary order," vide Durfee, *Amer. Jour. of Math.*, Vol. V, p. 349). The rule for modifying the process is to divide the horizontal row and corresponding vertical column by the coefficient of the leading separation. Observe that we thus obtain each leading term with coefficient unity, in itself a considerable advantage.

The inverse tables are further modified by multiplication throughout by some number which will get rid of fractions.

It will be convenient at this point to have before us the complete tables for the first six weights.

Tables of Symmetric Functions.

Weight 1. Partition (1).

	(1)
(1)	1

Weight 2. Partition (1^2).

	(2)	(1^2)
(1^2)		1
$(1)^2$	1	2

	(1^2)	$(1)^2$
(2)	$\bar{2}$	1
(1^2)	1	

Weight 2. Partition (2),

	(2)
(2)	1

Weight 3. Partition (1^3).

	(3)	(21)	(1^3)
(1^3)			1
(1^2)(1)		1	3
$(1)^3$	1	3	6

	(1^3)	(1^2)(1)	$(1)^3$
(3)	3	$\bar{3}$	1
(21)	$\bar{3}$	1	
(1^3)	1		

Weight 3. Partition (21).

	(3)	(21)
(21)		1
(2)(1)	1	1

	(21)	(2)(1)
(3)	$\bar{1}$	1
(21)	1	

Weight 3. Partition (3).

	(3)
(3)	1

Weight 4. Partition (1^4).

	(4)	(31)	(2^2)	(21^2)	(1^4)
(1^4)					1
$(1^3)(1)$				1	4
$(1^2)^2$			1	2	6
$(1^2)(1)^2$		1	2	5	12
$(1)^4$	1	4	6	12	24

	(1^4)	$(1^3)(1)$	$(1^2)^2$	$(1^2)(1)^2$	$(1)^4$
(4)	$\bar{4}$	4	2	$\bar{4}$	1
(31)	4	$\bar{1}$	$\bar{2}$	1	
(2^2)	2	$\bar{2}$	1		
(21^2)	$\bar{4}$	1			
(1^4)	1				

Weight 4. Partition (21^2).

	(4)	(31)	$2(2^2)$	(21^2)
(21^2)				1
$(21)(1)$		1	1	2
$(2)(1^2)$		1		1
$(2)(1)^2$	1	2	1	2

	(21^2)	$(21)(1)$	$(2)(1^2)$	$(2)(1)^2$
(4)	1	$\bar{1}$	$\bar{1}$	1
(31)	$\bar{1}$		1	
$2(2^2)$	$\bar{1}$	1	$\bar{1}$	
(21^2)	1			

Weight 4. Partition (2^2).

	(4)	(2^2)
(2^2)		1
$(2)^2$	1	2

	(2^2)	$(2)^2$
(4)	$\bar{2}$	1
(2^2)	1	

Weight 4. Partition (31).

	(4)	(31)
(31)		1
$(3)(1)$	1	1

	(31)	$(3)(1)$
(4)	$\bar{1}$	1
(31)	1	

Weight 4. Partition (4).

	(4)
(4)	1

WEIGHT 5. PARTITION (1^5).

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
(1^5)							1
$(1^4)(1)$						1	5
$(1^3)(1^2)$					1	3	10
$(1^3)(1)^2$				1	2	7	20
$(1^2)^2(1)$			1	2	5	12	30
$(1^2)(1)^3$		1	3	7	12	27	60
$(1)^5$	1	5	10	20	30	60	120

	(1^5)	$(1^4)(1)$	$(1^3)(1^2)$	$(1^3)(1)^2$	$(1^2)^2(1)$	$(1^2)(1)^3$	$(1)^5$
(5)	5	$\overline{5}$	$\overline{5}$	5	5	$\overline{5}$	1
(41)	$\overline{5}$	1	5	$\overline{1}$	$\overline{3}$	1	
(32)	$\overline{5}$	5	$\overline{1}$	$\overline{2}$	1		
(31^2)	5	$\overline{1}$	$\overline{2}$	1			
(2^21)	5	$\overline{3}$	1				
(21^3)	$\overline{5}$	1					
(1^5)	1						

WEIGHT 5. PARTITION (21^3).

	(5)	(41)	(32)	(31^2)	$2(2^21)$	(21^3)
(21^3)						1
$(21^2)(1)$				1	1	3
$(21)(1^2)+(1^3)(2)$			1	3	1	4
$(21)(1)^2$		1	3	4	3	6
$(2)(1^2)(1)$		1	1	3	1	3
$(2)(1)^3$	1	3	4	6	3	6

	(21^3)	$(21^2)(1)$	$(21)(1^2)+(1^3)(2)$ $\div 5$	$(21)(1)^2$	$(2)(1^2)(1)$	$(2)(1)^3$
(5)	$\overline{5}$	5	5	$\overline{5}$	$\overline{10}$	5
(41)	5		$\overline{5}$		5	
(32)	5	$\overline{5}$	1	2	$\overline{2}$	
(31^2)	$\overline{5}$		2	$\overline{1}$	1	
$2(2^21)$	$\overline{10}$	5	$\overline{2}$	1	$\overline{1}$	
(21^3)	5					

WEIGHT 5. PARTITION (2^21).

	(5)	(41)	(32)	(2^21)
(2^21)				1
$(2^2)(1)$			1	2
$(21)(2)$		1	1	2
$(2)^2(1)$	1	1	2	2

	(2^21)	$(2^2)(1)$	$(21)(2)$	$(2)^2(1)$
(5)	1	$\overline{1}$	$\overline{1}$	1
(41)	$\overline{1}$	$\overline{1}$	1	
(32)	$\overline{1}$	1		
(2^21)	1			

Weight 5. Partition (31^2).

	(5)	(41)	(32)	(31^2)
$(31)^2$				1
(31) (1)		1	1	2
(3) (1^2)		1		1
(3)(1^2)	1	2	1	2

	(31^2)	(31)(1)	(3)(1^2)	$(3)(1)^2$
(5)	1	$\bar{1}$	$\bar{1}$	1
(41)	$\bar{1}$		1	
(32)	$\bar{1}$	1	$\bar{1}$	
(31^2)	1			

Weight 5. Partition (32).

	(5)	(32)
(32)		1
(3)(2)	1	1

	(32)	(3)(2)
(5)	$\bar{1}$	1
(32)	1	

Weight 5. Partition (41).

	(5)	(41)
(41)		1
(4)(1)	1	1

	(41)	(4)(1)
(5)	$\bar{1}$	1
(41)	1	

Weight 5. Partition (5).

	(5)
(5)	1

Weight 6. Partition (1^6).

	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)
(1^6)											1
$(1^5)(1)$										1	6
$(1^4)(1^2)$									1	4	15
$(1^3)^2$								1	2	6	20
$(1^4)(1)^2$							1		2	9	30
$(1^3)(1^2)(1)$						1	3	3	8	22	60
$(1^3)(1)^3$					1	3	10	6	18	48	120
$(1^2)^3$				1		3	6	6	15	36	90
$(1^2)^2(1)^2$			1	2	2	8	18	15	34	78	180
$(1^2)(1)^4$		1	4	6	9	22	48	36	78	168	360
$(1)^6$	1	6	15	20	30	60	120	90	180	360	720

	(1^6)	$(1^5)(1)$	$(1^4)(1^2)$	$(1^3)^2$	$(1^4)(1)^2$	$(1^3)(1^2)(1)$	$(1^3)(1)^3$	$(1^2)^3$	$(1^2)^2(1)^2$	$(1^2)(1)^4$	$(1)^6$
(6)	$\overline{6}$	6	6	3	$\overline{6}$	$\overline{12}$	6	$\overline{2}$	9	$\overline{6}$	1
(51)	6	$\overline{1}$	$\overline{6}$	$\overline{3}$	1	7	$\overline{1}$	2	$\overline{4}$	1	
(42)	6	$\overline{6}$	2	$\overline{3}$	2	4	$\overline{2}$	$\overline{2}$	1		
(3^2)	3	$\overline{3}$	$\overline{3}$	3	3	$\overline{3}$		1			
(41^2)	$\overline{6}$	1	2	3	$\overline{1}$	$\overline{3}$	1				
(321)	$\overline{12}$	7	4	$\overline{3}$	$\overline{3}$	1					
(31^3)	6	$\overline{1}$	$\overline{2}$		1						
(2^3)	$\overline{2}$	2	$\overline{2}$	1							
(2^21^2)	9	$\overline{4}$	1								
(21^4)	$\overline{6}$	1									
(1^6)	1										

WEIGHT 6. PARTITION (21^4)

	(6)	(51)	(42)	$2\,(3^2)$	(41^2)	(321)	(31^3)	$3\,(2^3)$	$2\,(2^21^2)$	(21^4)
(21^4)										1
$(21^3)(1)$							1		1	4
$(21^2)(1^2)+(1^4)(2)$						1	4	1	2	7
$(21)(1^3)$						1	3		1	4
$(21^2)(1)^2$					1	3	6	2	5	12
$(21)(1^2)(1)+(1^3)(2)(1)$			1	1	3	6	13	2	6	16
$(21)(1)^3$		1	4	3	6	13	18	6	12	24
$(2)(1^2)^2$			1		2	2	6		2	6
$(2)(1^2)(1)^2$		1	2	1	5	6	12	2	5	12
$(2)(1)^4$	1	4	7	4	12	16	24	6	12	24

$\div 30.$

	(21^4)	$(21^3)(1)$	$(21^2)(1^2)+(1^4)(2)$	$(21)(1^3)$	$(21^2)(1)^2$	$(21)(1^2)(1)+(1^3)(2)(1)$	$(21)(1)^3$	$(2)(1^2)^2$	$(2)(1^2)(1)^2$	$(2)(1)^4$
(6)	30	$\overline{30}$	$\overline{30}$	$\overline{30}$	30	60	$\overline{30}$	30	$\overline{90}$	30
(51)	$\overline{30}$		30	30		$\overline{30}$		$\overline{30}$	30	
(42)	$\overline{30}$	30	$\overline{10}$	30	$\overline{10}$	$\overline{20}$	10	30	$\overline{10}$	
$2\,(3^2)$	$\overline{30}$	30	30	$\overline{60}$	$\overline{30}$	30		$\overline{30}$		
(41^2)	30		$\overline{10}$	$\overline{30}$	5	10	$\overline{5}$		5	
(321)	60	$\overline{30}$	$\overline{20}$	30	10	$\overline{4}$	2		$\overline{2}$	
(31^3)	$\overline{30}$		10		$\overline{5}$	2	$\overline{1}$		1	
$3\,(2^3)$	30	$\overline{30}$	30	$\overline{30}$						
$2\,(2^21^2)$	$\overline{90}$	30	$\overline{10}$		5	$\overline{2}$	1		$\overline{1}$	
(21^4)	30									

WEIGHT 6. PARTITION (2^21^2). $\div 15$.

	(6)	(51)	(42)	(3^2)	(41^2)	(321)	$3(2^3)$	(2^21^2)
(2^21^2)								1
$(2^21)(1)$						1	1	2
$(21^2)(2)+(2^2)(1^2)$				1	1	2		3
$(21)^2$			1	2	2	2	2	4
$(2^2)(1)^2$			1	2		2	1	2
$(21)(2)(1)$		1	2	2	2	3	2	4
$(2)^2(1^2)$		1		2	1	2		2
$(2)^2(1)^2$	1	2	3	4	2	4	2	4

	(2^21^2)	$(2^21)(1)$	$(21^2)(2)+(2^2)(1^2)$	$(21)^2$	$(2^2)(1)^2$	$(21)(2)(1)$	$(2)^2(1^2)$	$(2)^2(1)^2$
(6)	$\bar{10}$	10	10	5	$\bar{10}$	$\bar{20}$	$\bar{10}$	15
(51)	10	5	$\bar{10}$	$\bar{5}$	$\bar{5}$	5	10	
(42)	10	$\bar{10}$	2	$\bar{5}$	4	8	$\bar{8}$	
(3^2)	5	$\bar{5}$	$\bar{5}$	5	5	$\bar{5}$	5	
(41^2)	$\bar{10}$	$\bar{5}$	4	5	$\bar{7}$	1	$\bar{1}$	
(321)	$\bar{20}$	5	8	$\bar{5}$	1	2	$\bar{2}$	
$3(2^3)$	$\bar{10}$	10	$\bar{8}$	5	$\bar{1}$	$\bar{2}$	2	
(2^21^2)	15							

WEIGHT 6. PARTITION (2^3).

	(6)	(42)	(2^3)
(2^3)			1
$(2^2)(2)$		1	3
$(2)^3$	1	3	6

	(2^3)	$(2^2)(2)$	$(2)^3$
(6)	3	$\bar{3}$	1
(42)	$\bar{3}$	1	
(2^3)	1		

WEIGHT 6. PARTITION (31^3).

	(6)	(51)	$2(3^2)$	(42)	(41^2)	(321)	(31^3)
(31^3)							1
$(31^2)(1)$					1	1	3
$(3)(1^3)$					1		1
$(31)(1^2)$				1	2	1	3
$(31)(1)^2$		1	1	2	4	3	6
$(3)(1^2)(1)$		1		1	3	1	3
$(3)(1)^3$	1	3	1	3	6	3	6

	(31^3)	$(31^2)(1)$	$(3)(1^3)$	$(31)(1^2)$	$(31)(1)^2$	$(3)(1^2)(1)$	$(3)(1)^3$
(6)	$\bar{1}$	1	1	1	$\bar{1}$	$\bar{2}$	1
(51)	1		$\bar{1}$	$\bar{1}$		1	
$2(3^2)$	1	$\bar{1}$	2	$\bar{1}$	1	$\bar{1}$	
(42)	1	$\bar{1}$	$\bar{1}$	1			
(41^2)	$\bar{1}$		1				
(321)	$\bar{2}$	1	$\bar{1}$				
(31^3)	1						

WEIGHT 6. PARTITION (321). $\div 2$.

	(6)	(51)	(42)	$2(3^2)$	(321)
(321)					1
(32)(1)			1	1	1
(31)(2)		1		1	1
(3)(21)		1	1		1
(3)(2)(1)	1	1	1	1	1

	(321)	(32)(1)	(31)(2)	(3)(21)	(3)(2)(1)
(6)	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	2
(51)	$\bar{1}$	$\bar{1}$	1	1	
(42)	$\bar{1}$	1	$\bar{1}$	1	
$2(3^2)$	$\bar{1}$	1	1	$\bar{1}$	
(321)	1				

WEIGHT 6. PARTITION (3^2).

	(6)	(3^2)
(3^2)		1
$(3)^2$	1	2

	(3^2)	$(3)^2$
(6)	$\bar{2}$	1
(3^2)	1	

WEIGHT 6. PARTITION (41^2).

	(6)	(51)	(42)	(41^2)
(41^2)				1
(41)(1)		1	1	2
$(4)(1^2)$		1		1
$(4)(1)^2$	1	2	1	2

	(41^2)	(41)(1)	$(4)(1^2)$	$(4)(1)^2$
(6)	1	$\bar{1}$	$\bar{1}$	1
(51)	$\bar{1}$		1	
(42)	$\bar{1}$	1	$\bar{1}$	
(41^2)	1			

WEIGHT 6. PARTITION (42).

	(6)	(42)
(42)		1
(4)(2)	1	1

	(42)	(4)(2)
(6)	$\bar{1}$	1
(42)	1	

WEIGHT 6. PARTITION (51).

	(6)	(51)
(51)		1
(5)(1)	1	1

	(51)	(5)(1)
(6)	$\overline{1}$	1
(51)	1	

WEIGHT 6. PARTITION (6).

	(6)
(6)	1

In these tables I have slightly varied the order of the partitions as I found it convenient in different cases; the proper order of the partitions for a table of given partitions remains, I think, to be discovered.

The sums of the powers of the roots may be at once written down in terms of the table separations by means of an extension of Waring's formula which I have previously given (loc. cit.), and it is convenient to repeat it here. If

$$(\lambda^l\mu^m\dots\dots)$$

be any partition of n, and

$$(J_1)^{j_1}(J_2)^{j_2}\dots\dots$$

one of its separations, then

$$(-)^{l+m+\dots\dots}\frac{(l+m+\dots\dots-1)!}{l!\,m!\dots\dots}S_n$$
$$=\sum(-)^{j_1+j_2+\dots\dots}\frac{(j_1+j_2+\dots\dots-1)!}{j_1!\,j_2!\dots\dots}(J_1)^{j_1}(J_2)^{j_2}\dots\dots,$$

where it will be observed that any coefficient depends merely upon the multiplicity of the separation which it multiplies, and that the right-hand side is a function of the same assemblages of separations as are employed, by rule, in the tables. This very important result is a good example of the extent and scope of the new theory; the mere existence of so suggestive a formula, of which the simplicity vies with the generality, points with a clearness which cannot be mistaken, to the conclusion that up to the present the algebraic theory of symmetric functions has been regarded from a point of view so little elevated that its chief beauties remained obscured. It is convincing proof that much may be expected from a comprehensive survey from the new vantage point.

General Remarks upon Syzygies.

In general the expression of any symmetric function in terms of the table separations is unique, but where we are not restricted to these combinations this is not so. There are as many table separations as there are species partitions of those separations, and in general the whole number of separations exceeds the number of corresponding species partitions; this difference indicates the number of syzygies which exist between the separations. Thus, turning to the table of partition (21^4) we observe 10 species partitions and 12 separations pointing to $12 - 10 = 2$ syzygies between the separations; the actual forms of the syzygies are easily obtained, for consider the product

$$(21^2)(2),$$

we may *either* express (21^2) by means of separations of (1^4) *or* (2) by means of separations of (1^2); in either way we arrive at the expression of the product by means of separations of (21^4). Thus

$$\begin{aligned}(21^2)(2) &= \{(1^3)(1) - 4(1^4)\}(2)\\ &= (21^2)\{(1)^2 - 2(1^2)\}\end{aligned}$$

leading to the syzygy

$$2(21^2)(1^2) - (21^2)(1)^2 + (1^3)(2)(1) - 4(2)(1^4) = 0;$$

the other syzygy is obtained from the product

$$(21)(2)(1)$$

and is found to be

$$2(21)(1^2)(1) - 3(1^3)(2)(1) - (21)(1)^3 + (2)(1^2)(1)^2 = 0.$$

More conveniently we may regard the first of these syzygies as arising from the partition (42), for taking the product $(4)(2)$, we may simultaneously express (4) in terms of separations of (1^4) and (2) in terms of separations of (2), *or* simultaneously (4) in terms of separations of (21^2) and (2) in terms of separations of (1^2).

In a similar manner the second syzygy arises from the partition

$$(321),$$

where observe that the partitions

$$(42),$$
$$(321)$$

are the only ones containing two different parts besides unity. So, considering the separations of

$$(21^{n-2})$$

and

$$\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots\lambda_s^{l_s}1^i),$$

any one of its species partitions, we see that corresponding thereto there must be

$$s-1$$

syzygies, which may be written down according to the method above explained. The number of species partitions is the whole number of partitions less one, the generating function being

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots}-\frac{1}{1-x}.$$

If we write down all the partitions of a number and find that p_2 partitions contain a part 2, p_3 partitions a part 3, and so on, the number

$$p_2+p_3+\ldots\ldots+p_n$$

indicates the number of separations of

$$(21^{n-2});$$

the generating function for this number is

$$\frac{x^2}{(1-x)^2(1-x^2)(1-x^3)\ldots\ldots},$$

and hence the generating function for the syzygies between the separations of

$$(21^{n-2})$$

is

$$\frac{x^2}{(1-x)^2(1-x^2)(1-x^3)\ldots\ldots}-\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots}+\frac{1}{1-x},$$

which is

$$\frac{-1+x+x^2+(1-x)(1-x^2)(1-x^3)\ldots\ldots}{(1-x)^2(1-x^2)(1-x^3)\ldots\ldots},$$

or

$$\frac{\sum_{j=2}^{j=\infty}(-)^j(1+x^j)\,x^{\frac{1}{2}(3j^2-j)}}{(1-x)^2(1-x^2)(1-x^3)\ldots\ldots}.$$

The syzygies between the separations of other partitions may be worked out in a similar manner, though the calculations soon become laborious.

All these syzygies are linear relations between the separations of partitions, each syzygy involving only the separations of a single partition.

Beyond these there are syzygies connecting separations of different partitions which are at once seen by comparing the different tables of a given weight.

As regards any one table which expresses monomial symmetric functions in terms of separations of a given partition, any one row exhibits either such monomials as reducible quâ the separations, or else a congruence between the separations of highest degree.

By a comparison of the expressions of the same monomial symmetric function in different tables, syzygies are obtained between separations of different partitions; this circumstance I hope to discuss at some future time when applying the present theory to that of the covariants of binary forms.

Analogue of Newton's Theorem of the Sums of Powers.

The general expression of s_n in terms of the table separations of any partition of n was obtained by a comparison with a known particular case; for the proper discussion of the extended theory of symmetric functions brought forward here, it is necessary to see how the formula arises in another manner. We start with Newton's theorem for the expression of the sums of the powers in terms of the elementary coefficient, which is usually written as a series of relations, viz:

$$
\begin{aligned}
&s_1-(1)=0,\\
&s_2-(1)s_1+2(1^2)=0,\\
&s_3-(1)s_2+(1^2)s_1-3(1^3)=0,\\
&s_4-(1)s_3+(1^2)s_2-(1^3)s_1+4(1^4)=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

enabling the successive calculations of $s_2, s_3, s_4, \ldots.$ These relations are all exhibited by the single identity

$$\frac{(1)x-2(1^2)x^2+3(1^3)x^3-\ldots.}{1-(1)x+(1^2)x^2-(1^3)x^3+\ldots.}=s_1x+s_2x^2+s_3x^3+\ldots.,$$

for, on clearing of fractions and equating coefficients of like powers of x, the set of relations is produced.

Here, it is very important to observe, we can immediately obtain Waring's summation formula for the sums of the powers by expanding the denominator of the left-hand side by the multinomial theorem. This fact seems to have hitherto escaped the notice of writers upon the subject. This formula thus represents the expression of s_n in terms of separations of (1^n); the object in view is the corresponding formula for any other partition of n.

We can at once solve the problem for the partitions

$$(21^{n-2}),\ (31^{n-3}),\ (41^{n-4}),\ \ldots .\ (\lambda 1^{n-\lambda})\ \ldots .,$$

for, omitting the first of the series of relations, we may write

$$\begin{aligned}
&s_2 - (2) = 0, \\
&s_3 - (1)\,s_2 + (21) = 0, \\
&s_4 - (1)\,s_3 + (1^2)\,s_2 - (21^2) = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

a series which leads to the identity

$$\frac{(2)\,x^2 - (21)\,x^3 + (21^2)\,x^4 - \ldots .}{1 - (1)\,x + (1^2)\,x^2 - (1^3)\,x^3 + \ldots .} = s_2x^2 + s_3x^3 + s_4x^4 + \ldots .,$$

and now expanding the sinister by the multinomial theorem we necessarily exhibit s_n in terms of separations of (21^{n-2}) and arrive at the corresponding summation formula.

Omitting the first of the last set of relations, we may write

$$\begin{aligned}
&s_3 - (3) = 0, \\
&s_4 - (1)\,s_3 + (31) = 0, \\
&s_5 - (1)\,s_4 + (1^2)\,s_3 - 31^2 = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

and thence

$$\frac{(3)\,x^3 - (31)\,x^4 + (31^2)\,x^5 - \ldots .}{1 - (1)\,x + (1^2)\,x^2 - (1^3)\,x^3 + \ldots .} = s_3x^3 + s_4x^4 + s_5x^5 + \ldots .,$$

and proceeding in a similar manner we get finally

$$\frac{(\lambda)\,x^\lambda - (\lambda 1)\,x^{\lambda+1} + (\lambda 1^2)\,x^{\lambda+2} - \ldots .}{1 - (1)\,x + (1^2)\,x^2 - (1^3)\,x^3 + \ldots .} = s_\lambda x^\lambda + s_{\lambda+1}x^{\lambda+1} + s_{\lambda+2}x^{\lambda+2} + \ldots .,$$

a formula which enables us to exhibit s_n by means of separations of $(\lambda 1^{n-\lambda})$. Put now

$$1 - (1)\,x + (1^2)\,x^2 - (1^3)\,x^3 + \ldots . = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)\ \ldots .$$

and

$$\sum\nolimits'_{\lambda} = \sum \frac{\alpha^\lambda x^\lambda}{1 - \alpha x} = s_\lambda x^\lambda + s_{\lambda+1}x^{\lambda+1} + s_{\lambda+2}x^{\lambda+2} + \ldots .,$$

$$\sum\nolimits''_{\lambda} = \sum \frac{\alpha^\lambda x^\lambda}{(1 - \alpha x)^2} = s_\lambda x^\lambda + 2s_{\lambda+1}x^{\lambda+1} + 3s_{\lambda+2}x^{\lambda+2} + \ldots .,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sum\nolimits^{(m)}_{\lambda} = \sum \frac{\alpha^\lambda x^\lambda}{(1 - \alpha x)^m} = s_\lambda x^\lambda + \frac{1}{2!}\,m\,(m+1)\,s_{\lambda+1}x^{\lambda+1}$$

$$+ \frac{1}{3!}\,m\,(m+1)(m+2)\,s_{\lambda+2}x^{\lambda+2} + \ldots .,$$

we have

$$\sum\nolimits'_{\lambda}\sum\nolimits'_{\mu}-\sum\nolimits''_{\lambda+\mu}=\sum\frac{\alpha^{\lambda}x^{\lambda}}{1-\alpha x}\sum\frac{\alpha^{\mu}x^{\mu}}{1-\alpha x}-\sum\frac{\alpha^{\lambda+\mu}x^{\lambda+\mu}}{(1-\alpha x)^{2}}$$

$$=\frac{\sum\alpha^{\lambda}\beta^{\mu}x^{\lambda+\mu}(1-\gamma x)(1-\delta x)}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)\,\ldots\ldots}$$

$$=\frac{(\lambda\mu)\,x^{\lambda+\mu}-(\lambda\mu1)\,x^{\lambda+\mu+1}+(\lambda\mu1^{2})\,x^{\lambda+\mu+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-(1^{3})\,x^{3}+\ldots\ldots},$$

or, taking previous results into account, this identity may be written

$$\frac{(\lambda)\,x^{\lambda}-(\lambda1)\,x^{\lambda+1}+(\lambda1^{2})\,x^{\lambda+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\mu)\,x^{\mu}-(\mu1)\,x^{\mu+1}+(\mu1^{2})\,x^{\mu+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$-\frac{(\lambda\mu)\,x^{\lambda+\mu}-(\lambda\mu1)\,x^{\lambda+\mu+1}+(\lambda\mu1^{2})\,x^{\lambda+\mu+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$=s_{\lambda+\mu}x^{\lambda+\mu}+2s_{\lambda+\mu+1}x^{\lambda+\mu+1}+3s_{\lambda+\mu+2}x^{\lambda+\mu+2}+\ldots\ldots,$$

a formula which leads to the expression of s_n in terms of separations of $(\lambda\mu1^{n-\lambda-\mu})$. Proceeding, we find

$$\sum\nolimits'_{\lambda}\sum\nolimits'_{\mu}\sum\nolimits'_{\nu}-\sum\nolimits'_{\lambda}\sum\nolimits''_{\mu+\nu}-\sum\nolimits'_{\mu}\sum\nolimits''_{\lambda+\nu}-\sum\nolimits'_{\nu}\sum\nolimits''_{\lambda+\mu}+2\sum\nolimits'''_{\lambda+\mu+\nu}$$

$$=\frac{(\lambda\mu\nu)\,x^{\lambda+\mu+\nu}-(\lambda\mu\nu1)\,x^{\lambda+\mu+\nu+1}+(\lambda\mu\nu1^{2})\,x^{\lambda+\mu+\nu+2}-\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-(1^{3})\,x^{3}+\ldots\ldots},$$

which should be compared and contrasted with the ordinary symmetric function formula

$$s_{\lambda}s_{\mu}s_{\nu}-s_{\lambda}s_{\mu+\nu}-s_{\mu}s_{\lambda+\nu}-s_{\nu}s_{\lambda+\mu}+2s_{\lambda+\mu+\nu}=(\lambda\mu\nu).$$

We are now led to the formula

$$\frac{(\lambda)\,x^{\lambda}-(\lambda1)\,x^{\lambda+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\mu)\,x^{\mu}-(\mu1)\,x^{\mu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\nu)\,x^{\nu}-(\nu1)\,x^{\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$-\frac{1}{2}\,\frac{(\lambda)\,x^{\lambda}-(\lambda1)\,x^{\lambda+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\mu\nu)\,x^{\mu+\nu}-(\mu\nu1)\,x^{\mu+\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$-\frac{1}{2}\,\frac{(\mu)\,x^{\mu}-(\mu1)\,x^{\mu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\lambda\nu)\,x^{\lambda+\nu}-(\lambda\nu1)\,x^{\lambda+\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$-\frac{1}{2}\,\frac{(\nu)\,x^{\nu}-(\nu1)\,x^{\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}\cdot\frac{(\lambda\mu)\,x^{\lambda+\mu}-(\lambda\mu1)\,x^{\lambda+\mu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$+\frac{1}{2}\,\frac{(\lambda\mu\nu)\,x^{\lambda+\mu+\nu}-(\lambda\mu\nu1)\,x^{\lambda+\mu+\nu+1}+\ldots\ldots}{1-(1)\,x+(1^{2})\,x^{2}-\ldots\ldots}$$

$$=s_{\lambda+\mu+\nu}x^{\lambda+\mu+\nu}+3s_{\lambda+\mu+\nu+1}x^{\lambda+\mu+\nu+1}+6s_{\lambda+\mu+\nu+2}x^{\lambda+\mu+\nu+2}+\ldots\ldots,$$

enabling us to express s_n in terms of separations of

$$(\lambda\mu\nu 1^{n-\lambda-\mu-\nu}).$$

From the mode in which these results have been evolved, it is absolutely certain, without further demonstration, that the expression

$$\frac{(\lambda\mu\nu\ldots\ldots)\,x^{\lambda+\mu+\nu+\ldots\ldots} - (\lambda\mu\nu\ldots\ldots 1)\,x^{\lambda+\mu+\nu+\ldots\ldots+1} + \ldots\ldots}{1-(1)\,x+(1^2)\,x^2-\ldots\ldots}$$

is given in terms of

$$\sum\nolimits'_{\lambda}, \sum\nolimits'_{\mu}, \sum\nolimits'_{\nu}, \ldots\ldots \sum\nolimits''_{\lambda+\mu}, \sum\nolimits''_{\lambda+\nu}, \sum\nolimits''_{\mu+\nu}, \ldots\ldots \sum\nolimits'''_{\lambda+\mu+\nu}, \ldots\ldots$$

by precisely the same law as

$$(\lambda\mu\nu\ldots\ldots)$$

is given in terms of

$$s_\lambda,\ s_\mu,\ s_\nu,\ \ldots\ldots s_{\lambda+\mu},\ s_{\lambda+\nu},\ s_{\mu+\nu},\ \ldots\ldots s_{\lambda+\mu+\nu},\ \ldots\ldots$$

If any number of equalities exist between the numbers

$$\lambda,\ \mu,\ \nu,\ \ldots\ldots$$

the same modifications nre requisite in both systems of formulae; accordingly both systems are inverted in a similar manner and following the same law; thus the formula above written is obtained by a comparison with the formula

$$(\lambda)(\mu)(\nu) - \frac{1}{2}(\lambda)(\mu\nu) - \frac{1}{2}(\mu)(\lambda\nu) - \frac{1}{2}(\nu)(\lambda\mu) + \frac{1}{2}(\lambda\mu\nu) = s_{\lambda+\mu+\nu};$$

in this manner is established the identity which for any separable partition is the direct analogue of the elementary formula

$$\frac{(1)\,x - 2\,(1^2)\,x^2 + 3\,(1^3)\,x^3 - \ldots\ldots}{1-(1)\,x+(1^2)\,x^2-(1^3)\,x^3+\ldots\ldots} = s_1x + s_2x^2 + s_3x^3 + \ldots\ldots,$$

and this leads by multinomial expansions to a proof of the summation formula which expresses s_n in terms of separations of any partition of n.

Derivation of Symmetric Functions from the Sums of Powers.

In the ordinary theory we derive the symmetric functions $(\lambda\mu)$, $(\lambda\mu\nu)$, ... from the formulae

$$(\lambda\mu) = s_\lambda s_\mu - s_{(\lambda+\mu)} \equiv s\,(1^\lambda)\,s\,(1^\mu) - s\,(1^{\lambda+\mu}),^*$$

$$(\lambda\mu\nu) = s_\lambda s_\mu s_\nu - s_\lambda s_{\mu+\nu} - s_\mu s_{\nu+\lambda} - s_\nu s_{\lambda+\mu} + 2s_{\lambda+\mu+\nu},$$

$$\equiv s\,(1^\lambda)\,s\,(1^\mu)\,s\,(1^\nu) - s\,(1^\lambda)\,s\,(1^{\mu+\nu}) - s\,(1^\mu)\,s\,(1^{\nu+\lambda}) - s\,(1^\nu)\,s\,(1^{\lambda+\mu})$$

$$+ 2s\,(1^{\lambda+\mu+\nu}),$$

---

[*] N. B. $s\,(\lambda\mu\ldots\ldots)$ denotes the expression of $s_{\lambda+\mu+\ldots\ldots}$ by means of separations of $(\lambda\mu\ldots\ldots)$.

and no difficulty is presented, because the products

$$s(1^\lambda)\,s(1^\mu),$$
$$s(1^\lambda)\,s(1^\mu)\,s(1^\nu),\ s(1^\lambda)\,s(1^{\mu+\nu}),\ s(1^\mu)\,s(1^{\nu+\lambda}),\ s(1^\nu)\,s(1^{\lambda+\mu}),$$

are, by direct multiplication, obtained in terms of separations of $(1^{\lambda+\mu})$, $(1^{\lambda+\mu+\nu})$ respectively. Had we been concerned with a separate partition composed of dissimilar parts, we could not thus have proceeded by direct multiplication; we could indeed have obtained the products in terms of separations, but the process would not have been a unique one and we would not have obtained an expression involving the tabular assemblages of separations. It is this expression that is required. Take the separable partion

$$(\lambda 1^{n-\lambda});$$

supposing it to be possible to express the product

$$s_p s_{n-p}$$

in terms of its tabular separations, we have, in the first place, two alternatives (in general). We may express s_p in terms of separations of (1^p), and s_{n-p} in terms of separations of $(\lambda 1^{n-p-\lambda})$, or *vice versa;* but neither of the products

$$s(1^p)\,s(\lambda 1^{n-p-\lambda}),$$
$$s(\lambda 1^{p-\lambda})\,s(1^{n-p})$$

are in terms of tabular separations of $(\lambda 1^{n-\lambda})$; we have in fact to take a linear function of these products. The investigation is facilitated by proceeding at once to the general case.

Let us then consider the problem of expressing the product

$$s_\lambda^l s_\mu^m \ldots .$$

by means of tabular separations of the partition

$$(t_1^{\tau_1} t_2^{\tau_2} t_3^{\tau_3} \ldots .).$$

Suppose $\qquad (\lambda_1^{l_1}\mu_1^{m_1} \ldots .)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2} \ldots .)^{L_2} \ldots .$
to be any separation of $\qquad (t_1^{\tau_1} t_2^{\tau_2} t_3^{\tau_3} \ldots .),$
having the species partition

$$(\lambda^l \mu^m \ldots .);$$

the general summation formula gives us

$$s(\lambda_1^{l_1}\mu_1^{m_1} \ldots .) = \ldots . + (-)^{l_1+m_1+\ldots+1}\frac{l_1!\ m_1! \ldots .}{(l_1+m_1+\ldots .-1)!}(\lambda_1^{l_1}\mu_1^{m_1} \ldots .);$$

hence, observing that

$$L_1(l_1+m_1+\ldots.)+L_2(l_2+m_2+\ldots.)+\ldots.=\tau_1+\tau_2+\tau_3+\ldots.$$

we find

$$\{s(\lambda_1^{l_1}\mu_1^{m_1}\ldots.)\}^{L_1}\{s(\lambda_2^{l_2}\mu_2^{m_2}\ldots.)\}^{L_2}\ldots.$$
$$=\ldots.+(-)^{\tau_1+\tau_2+\ldots.+L_1+L_2+\ldots.}\left\{\frac{l_1!\ m_1!\ldots.}{(l_1+m_1+\ldots.-1)!}\right\}^{L_1}$$
$$\left\{\frac{l_2!\ m_2!\ldots.}{(l_2+m_2+\ldots.-1)!}\right\}^{L_2}\ldots.(\lambda_1^{l_1}\mu_1^{m_1}\ldots.)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\ldots.)^{L_2}\ldots.$$

Taking the summation for every separation having the species partition

$$(\lambda^l\mu^m\ldots.),$$

we may write

$$\sum\left\{\frac{(l_1+m_1+\ldots.-1)!}{l_1!\ m_1!\ldots.}\right\}^{L_1}\left\{\frac{(l_2+m_2+\ldots.-1)!}{l_2!\ m_2!\ldots.}\right\}^{L_2}\ldots.\frac{(L_1+L_2+\ldots.-1)!}{L_1!\ L_2!\ldots.}$$
$$\{s(\lambda_1^{l_1}\mu_1^{m_1}\ldots.)\}^{L_1}\{s(\lambda_2^{l_2}\mu_2^{m_2}\ldots.)\}^{L_2}\ldots.$$
$$=\ldots.+\sum(-)^{\tau_1+\tau_2+\ldots.+L_1+L_2+\ldots.}\frac{(L_1+L_2+\ldots.-1)!}{L_1!\ L_2!\ldots.}$$
$$(\lambda_1^{l_1}\mu_1^{m_1}\ldots.)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\ldots.)^{L_2}\ldots.$$

Observe that the dexter of this identity agrees with the corresponding portion of the expression of $s(t_1^{\tau_1}t_2^{\tau_2}\ldots.)$, for

$$s(t_1^{\tau_1}t_2^{\tau_2}\ldots.)=\ldots.$$
$$+\sum(-)^{\tau_1+\tau_2+\ldots+L_1+L_2+\ldots}\frac{(L_1+L_2+\ldots-1)!}{L_1!\ L_2!\ldots}(\lambda_1^{l_1}\mu_1^{m_1}\ldots)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\ldots)^{L_2}\ldots+\ldots,$$

and that we cannot obtain this agreement unless we take the linear function of the s products which appears on the sinister side; but we know that the s product

$$s_\lambda^l s_\mu^m\ldots.$$

is necessarily expressible by means of tabular separations, because each of the monomial forms, which arise on performing the multiplication, is so; hence the above linear function of the s products must be a function of tabular separations of the partition $(t_1^{\tau_1}t_2^{\tau_2}\ldots.)$.

We thus arrive at the result:

$$s_\lambda^l s_\mu^m\ldots.=\frac{\displaystyle\sum\left\{\frac{(l_1+m_1+\ldots.-1)!}{l_1!\ m_1!\ldots.}\right\}^{L_1}\left\{\frac{(l_2+m_2+\ldots.-1)!}{l_2!\ m_2!\ldots.}\right\}^{L_2}\ldots.\ \frac{(L_1+L_2+\ldots.-1)!}{L_1!\ L_2!\ldots.}\{s(\lambda_1^{l_1}\mu_1^{m_1}\ldots.)\}^{L_1}\{s(\lambda_2^{l_2}\mu_2^{m_2}\ldots.)\}^{L_2}\ldots.}{\displaystyle\sum\left\{\frac{(l_1+m_1+\ldots.-1)!}{l_1!\ m_1!\ldots.}\right\}^{L_1}\left\{\frac{(l_2+m_2+\ldots.-1)!}{l_2!\ m_2!\ldots.}\right\}^{L_2}\ldots.\frac{(L_1+L_2+\ldots.-1)!}{L_1!\ L_2!\ldots.}}$$

where

(i) the separable partition is $(t_1^{\tau_1}t_2^{\tau_2})$,

(ii) $(\lambda_1^{l_1}\mu_1^{m_1}\ldots.)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\ldots.)^{L_2}\ldots.$ is any separation having the species partition $(\lambda^l\mu^m\ldots.)$,

(iii) the summations are in regard to all such separations; the result being that the dexter side is a function of tabular separations of $(t_1^{\tau_1}t_2^{\tau_2}\ldots.)$.

I illustrate the process by the calculation of (42) in terms of separations of $(2^2 1^2)$. We have $$(42) = s_4 s_2 - s_6,$$
and

$$s_4 s_2 = \frac{\frac{1}{2}\cdot\frac{1}{2}\,s(2^2)\,s(1^2) + s(21^2)\,s(2)}{\frac{1}{2}\cdot\frac{1}{2} + 1} = \frac{1}{5}\,s(2^2)\,s(1^2) + \frac{4}{5}\,s(21^2)\,s(2)$$

$$= \frac{1}{5}\{-2(2^2)+(2)^2\}\{-2(1^2)+(1)^2\} + \frac{4}{5}\{(21^2)-(21)(1)-(2)(1^2)+(2)(1)^2\}(2),$$

$$= \frac{4}{5}(21^2)(2) + \frac{4}{5}(2^2)(1^2) - \frac{2}{5}(2^2)(1)^2 - \frac{6}{5}(2)^2(1^2) + (2)^2(1)^2,$$

which, since the separations $(21^2)(2)$, $(2^2)(1^2)$ occur with the same coefficient $\frac{4}{5}$, is, as it should be, a function of tabular separations.

Hence, extracting the value of $s_6 = s(2^2 1^2)$ from the tables, we find

$$\begin{aligned} 15(42) = {} & 10(2^2 1^2) - 10(2^2 1)(1) + 2\{(21^2)(2) + (2^2)(1^2)\} - 5(21)^2 \\ & + 4(2^2)(1)^2 + 8(21)(2)(1) - 8(2)^2(1^2). \end{aligned}$$

In this general way we are enabled, as in the ordinary theory, to combine the expressions for the sums of the powers so as to obtain the expressions of other symmetric functions by means of separations of any selected partition.

It will be gathered that this process would be very laborious in the case of symmetric functions whose partitions contained many parts. This is also the case in the ordinary or unitary theory (where by "unitary" it is meant that separable partitions composed wholly of parts, unity, are alone considered); it will be seen hereafter that this process will, for such forms, be naturally rejected in favor of easier methods of calculation.

Properties of Coefficients and Groups of Separations.

The general expression for the sum of the powers in terms of separations of any partition has, in regard to the numerical coefficients, a certain interesting and important property.

If the separable partition be (1^n) or (λ^n), it is a known theorem that the sum of the coefficients is

$$(-)^{n+1}.$$

Now, whenever the separable partition contains *dissimilar* parts, the sum of all the coefficients is *invariably* zero. This will be established by proving another theorem of a much more refined character from which it is immediately deducible. The separations of a given partition may be grouped in a manner which is independent of their species partitions. For clearness I take as a particular case the separable partition

$$(\lambda^3\mu^2)$$

and write down any one of its separations, say

$$(\lambda^2)(\lambda\mu)(\mu).$$

This separation may be regarded as compounded of the two separations:

$$(\lambda^2)(\lambda) \text{ of } (\lambda^3)$$

and

$$(\mu)^2 \text{ of } (\mu^2);$$

and moreover we will find that three other separations possess the same property, viz:

$$(\lambda^2)(\lambda)(\mu)^2,$$
$$(\lambda^2\mu)(\lambda)(\mu),$$
$$(\lambda^2\mu)(\lambda\mu);$$

for on suppressing the μ's in each we are left with

$$(\lambda^2)(\lambda),$$

and on suppressing the λ's there remains

$$(\mu)^2.$$

I say that these four separations

$$(\lambda^2)(\lambda\mu)(\mu),$$
$$(\lambda^2)(\lambda)(\mu)^2,$$
$$(\lambda^2\mu)(\lambda)(\mu),$$
$$(\lambda^2\mu)(\lambda\mu),$$

form a group, each member of which is compounded of the two separations

$$(\lambda^2)(\lambda),\ (\mu)^2.$$

I will call it the group $\{(\lambda^2)(\lambda),\ (\mu)^2\}$.

Consider now the separations of

$$(\lambda^3\mu^2)$$

as divided into six groups, viz:

$$\begin{array}{ll} \text{The group} & \{(\lambda)^3 \quad , (\mu)^2\}, \\ \text{``} & \{(\lambda)^3 \quad , (\mu^2)\}, \\ \text{``} & \{(\lambda^2)(\lambda), (\mu)^2\}, \\ \text{``} & \{(\lambda^2)(\lambda), (\mu^2)\}, \\ \text{``} & \{(\lambda^3) \quad , (\mu)^2\}, \\ \text{``} & \{(\lambda^3) \quad , (\mu^2)\}, \end{array}$$

where, be it observed, there is a group corresponding to every compound of a separation of (λ^3) with a separation of (μ^2).

So in general, if the separable partition be

$$(\lambda^l \mu^m \nu^n \ldots.),$$

we may take any separations of (λ^l), (μ^m), (ν^n) .... and form the group which is compounded of these separations. If (λ^l), (μ^m), (ν^n) .... possess L, M, N.... separations* respectively, there will thus be

$$LMN \ldots. \text{ groups.}$$

It will be observed that the grouping depends merely upon the multiplicities l, m, n, ... and not at all upon the parts λ, μ, ν, ...

I now enunciate the theorem:

Theorem: "In the expression of s_n in terms of separations of any partition of n, which contains dissimilar parts, the sum of the coefficients of the separations in each group is zero." I subjoin an example:

$$\begin{array}{rllll} \frac{3}{2} s(\lambda^2\mu^2) = & \frac{3}{2}(\lambda)^2(\mu)^2 & -(\lambda)^2(\mu^2) & +(\lambda^2)(\mu)^2 & +(\lambda^2)(\mu^2) \\ & -2(\lambda\mu)(\lambda)(\mu) & +(\lambda\mu^2)(\lambda) & +(\lambda^2\mu)(\mu) & -(\lambda^2\mu^2) \\ & +\frac{1}{2}(\lambda\mu)^2 & & & \\ \hline & \pm 2 & \pm 1 & \pm 1 & \pm 1 \\ \hline \end{array}$$

wherein the first, second, third, fourth columns of separations constitute the groups

$$\{(\lambda)^2, (\mu)^2\}, \ \{(\lambda)^2, (\mu^2)\}, \ \{(\lambda^2), (\mu)^2\}, \ \{(\lambda^2), (\mu^2)\}$$

respectively, and the sum of the coefficients of the separations in each group is seen to be zero.

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* Or, what is the same thing, if the numbers l, m, n .... possess L, M, N.... partitions respectively.

To establish this theorem, consider in the first place the identity

$$\frac{(\lambda)x^\lambda-(\lambda 1)x^{\lambda+1}+(\lambda 1^2)x^{\lambda+2}-\ldots\ldots}{1-(1)x+(1^2)x^2-(1^3)x^3+\ldots\ldots}=s_\lambda x^\lambda+s_{\lambda+1}x^{\lambda+1}+s_{\lambda+2}x^{\lambda+2}+\ldots\ldots;$$

write the left-hand side in the form

$$\frac{(\lambda)y_\lambda-(\lambda 1)y_\lambda x_1+(\lambda 1^2)y_\lambda x_2-(\lambda 1^3)y_\lambda x_3+\ldots\ldots}{1-(1)x_1+(1^2)x_2-(1^3)x_3+\ldots\ldots},$$

wherein y_λ, x_t are symbolical representations of

x^λ and x^t respectively.

If this expression be developed by the multinomial theorem, we obtain a product

$$y_\lambda x_1^{l_1}x_2^{l_2}x_3^{l_3}\ldots\ldots$$

multiplied by a certain symmetric function. This function consists of a number of symmetric functions, each of which is a separation of

$$(\lambda 1^{l_1+2l_2+3l_3+\ldots\ldots})$$

and each of which is a member of the group of separations

$$\{(\lambda),\ (1)^{l_1}(1^2)^{l_2}(1^3)^{l_3}\ldots\ldots\}.$$

The complete symmetric function which multiplies

$$y_\lambda x_1^{l_1}x_2^{l_2}x_3^{l_3}\ldots\ldots$$

constitutes those terms in the expression of

$$s_{\lambda+l_1+2l_2+3l_3+\ldots\ldots}$$

which belong to the group of separations

$$\{(\lambda),\ (1)^{l_1}(1^2)^{l_2}(1^3)^{l_3}\ldots\ldots\}.$$

Putting
$$(\lambda)=(\lambda 1)=(\lambda 1^2)=\ldots\ldots=1,$$
$$(1)=(1^2)=\ldots\ldots=1,$$

the left-hand side of the identity reduces to

$$y_\lambda;$$

hence, in the development by the multinomial theorem, all the other terms must vanish; but under these circumstances the aggregate of those separations which occur in any group of separations of a partition containing dissimilar parts, becomes the algebraic sum of the numerical coefficients which occur in such group; it follows that if s_n be expressed by means of separations of

$$(\lambda 1^{n-\lambda})\qquad n>\lambda$$

the algebraic sum of the coefficients in each group must be zero.

Next consider the two identities

$$\frac{(\lambda)x^{\lambda}-(\lambda 1)x^{\lambda+1}+\ldots.}{1-(1)x+(1^2)x^2-\ldots.}\cdot\frac{(\mu)x^{\mu}-(\mu 1)x^{\mu+1}+\ldots.}{1-(1)x+(1^2)x^2-\ldots.}-\frac{(\lambda\mu)x^{\lambda+\mu}-(\lambda\mu 1)x^{\lambda+\mu+1}+\ldots.}{1-(1)x+(1^2)x^2-\ldots.}$$

$$=s_{\lambda+\mu}x^{\lambda+\mu}+2s_{\lambda+\mu+1}x^{\lambda+\mu+1}+3s_{\lambda+\mu+2}x^{\lambda+\mu+2}+\ldots.,$$

$$\left\{\frac{(\lambda)x^{\lambda}-(\lambda 1)x^{\lambda+1}+\ldots.}{1-(1)x+(1^2)x^2-\ldots.}\right\}^2-2\,\frac{(\lambda^2)x^{2\lambda}-(\lambda^2 1)x^{2\lambda+1}+\ldots.}{1-(1)x+(1^2)x^2-\ldots.}$$

$$=s_{2\lambda}x^{2\lambda}+2s_{2\lambda+1}x^{2\lambda+1}+3s_{2\lambda+2}x^{2\lambda+2}+\ldots.$$

These may, as regards their left-hand sides, be written symbolically

$$\frac{(\lambda)y_{\lambda}-(\lambda 1)y_{\lambda}x_1+(\lambda 1^2)y_{\lambda}x_2-\ldots.}{1-(1)x_1+(1^2)x_2-(1^3)x_3+\ldots.}\cdot\frac{(\mu)z_{\mu}-(\mu 1)z_{\mu}x_1+(\mu 1^2)z_{\mu}x_2-\ldots.}{1-(1)x+(1^2)x_2-(1^3)x_3+\ldots.}$$

$$-\frac{(\lambda\mu)y_{\lambda}z_{\mu}-(\lambda\mu 1)y_{\lambda}z_{\mu}x_1+\ldots.}{1-(1)x+(1^2)x_2-\ldots.}\left\{\frac{(\lambda)y_{\lambda}-(\lambda 1)y_{\lambda}x_1+(\lambda 1^2)y_{\lambda}x_2+\ldots.}{1-(1)x_1+(1^2)x_2-(1^3)x_3+\ldots.}\right\}^2$$

$$-2\,\frac{(\lambda^2)y_{2\lambda}-(\lambda^2 1)y_{2\lambda}x_1+(\lambda^2 1^2)y_{2\lambda}x_2-\ldots.}{1-(1)x_1+(1^2)x_2-\ldots.},$$

and herein putting all the symmetric functions equal to unity, we obtain respectively

$$\text{zero},$$

and

$$y_{\lambda}^2-2y_{2\lambda};$$

hence, when s_n is expressed by means of separations of

$$(\lambda\mu 1^{n-\lambda-\mu}),$$

the algebraic sum of the coefficients in each group of separations is zero, and also when s_n is expressed by means of separations of

$$(\lambda^2 1^{n-2\lambda}) \qquad n>2\lambda$$

the algebraic sum of the coefficients in each group is zero also.

This method of proof is perfectly general and leads to the important conclusion that whenever s_n is expressed by means of separations of a partition, which does not consist merely of repetitions of a single part, the algebraic sum of the coefficients in each group is zero.

In the proof it has been assumed that the algebraic sum of the coefficients on the dexter of such relation as

$$s(\lambda\mu\nu)=(\lambda)(\mu)(\nu)-\tfrac{1}{2}(\lambda\mu)(\nu)-\tfrac{1}{2}(\lambda\nu)(\mu)-\tfrac{1}{2}(\mu\nu)(\lambda)+\tfrac{1}{2}(\lambda\mu\nu),$$

is zero. It is easily seen that this is so, for writing down the relations

$$\begin{aligned}(\lambda) &= s(\lambda),\\ (\lambda\mu) &= s(\lambda)\,s(\mu) - s(\lambda\mu),\\ \text{etc.} &= \text{etc.}\end{aligned}$$

if we put $(\lambda) = (\lambda\mu) = \ldots\ldots = 1$, we have

$$s(\lambda) = s(\mu) = 1,$$

and hence from the second relation

$$s(\lambda\mu) = 0.$$

Also the third relation being

$$(\lambda\mu\nu) = s(\lambda)\,s(\mu)\,s(\nu) - s(\lambda\mu)\,s(\nu) - s(\lambda\nu)\,s(\mu) - s(\mu\nu)\,s(\lambda) + 2s(\lambda\mu\nu),$$

putting

$$\begin{aligned}(\lambda\mu\nu) = 1,\ s(\lambda) = s(\mu) = s(\nu) = 1,\\ s(\lambda\mu) = s(\lambda\nu) = s(\mu\nu) = 0,\end{aligned}$$

we find $s(\lambda\mu\nu) = 0$, and so on in general, the expression for s_n becomes zero when, the separable partition containing dissimilar parts, the separates of the separations are each put equal to unity.

It is noteworthy that this property of the coefficients is not a generalization or extension of any known property, but is one which, by its nature, only comes into existence with the new theory; there is in fact no corresponding property in the ordinary theory, for in that theory the "group" does not exist *as a group*, but merely as an isolated term without interest. The theorem is of great importance as a verification of the coefficients of the separations, as well as for ascertaining that no separation has been accidentally omitted. It constitutes an example of departure from absolute generality.

This theory of the group may be easily extended to other symmetric functions.

Consider for a moment the expression of (3^2) by means of separations of (21^4). We have

$$2(3^2) = s_3^2 - s_6 = s(21)\,s(1^3) - s(21^4),$$

wherein we know that in both $s(21)$ and $s(21^4)$ the algebraic sum of the coefficients is zero; hence also in (3^2) the algebraic sum of all the coefficients must vanish; if, on the dexter side, there had been such a product as

$$s(1^4)\,s(2),$$

this could not have been the case, because the algebraic sum of the coefficients is not zero either in $s(1^4)$ or in $s(2)$. This s product does not occur simply because the partition (3^2) possesses no separation of species partition (42).

This reasoning is manifestly general, and we are thus led to the important theorem:

Theorem: "In the expression of symmetric function

$$(p_1^{\pi_1} p_2^{\pi_2} \ldots .)$$

by means of separations of

$$(t_1^{\tau_1} t_2^{\tau_2} \ldots .),$$

the algebraic sum of all the coefficients will be zero if the partition

$$(p_1^{\pi_1} p_2^{\pi_2} \ldots .)$$

possesses no separations of species partition

$$(\tau_1 t_1, \tau_2 t_2)."$$

The theorem may be verified from the tables in the cases of

Separable partition.		Symmetric functions.
(21^4)	for	(51), (3^2),
$(2^2 1^2)$	for	(51), (3^2),
(31^3)	for	(51), (42), (41^2),
(321)	for	(51), (42), (3^2),
(41^2)	for	(51).

The group coefficient theory obtains also under the same circumstances; to establish this it will suffice to show that if we take any two sums of powers, say

$$s(\lambda^5 \mu^4) \text{ and } s(\lambda^3 \mu^2),$$

multiply out their product and arrange the resulting separations in groups, the algebraic sum of the coefficients in each group is zero. In $s(\lambda^5 \mu^4)$ there is a group $\{(\lambda)^5, (\mu^2)(\mu)^2\}$, and in $s(\lambda^3 \mu^2)$ a group $\{(\lambda^2)(\lambda), (\mu^2)\}$,

and if we multiply the separations in these two groups together, we will, quâ the separable partitions $(\lambda^8 \mu^6)$, obtain separations belonging to the group

$$\{(\lambda)^6(\lambda^2),\ (\mu^2)^2(\mu)^2\}.$$

In this batch of separations the algebraic sum of the coefficients is necessarily zero; we obtain another batch of separations comprised in the *same* group by multiplication of the groups

$$\{(\lambda)^3(\lambda^2),\ (\mu^2)^2\} \text{ and } \{(\lambda)^3,\ (\mu)^2\},$$

and again for this batch of separations the algebraic sum of the coefficients is zero. Observe that by thus multiplying certain *complete* groups of the one expression by certain *complete* groups of the other, we must obtain all the separations of $(\lambda^8\mu^6)$ belonging to the group

$$\{(\lambda)^6(\lambda^2),\ (\mu^2)^2(\mu)^2\}$$

which occur in the product $s(\lambda^5\mu^4)\,s(\lambda^3\mu^2)$. Hence, if in such product there occurs one factor at least of which the separable partition does not consist merely of repetitions of a single part, the algebraic sum of the coefficients in each group of the product will be zero. We are thus enabled to enunciate as follows:

Theorem: "In the expression of symmetric function

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots.),$$

by means of separations of

$$(t_1^{\tau_1}t_2^{\tau_2}\ldots.),$$

the algebraic sum of the coefficients *in each group* will be zero, if the partition

$$(p_1^{\pi_1}p_2^{\pi_2}\ldots.)$$

possesses no separations of species partition

$$(\tau_1 t_1,\ \tau_2 t_2\ldots.)."$$

Concluding Remarks.

So far I have merely stood upon the threshold of the theory; the further development must be reserved for some future occasion; systems of partial differential equations exist analogous to those employed with such marked success by Brioschi, Hammond and others; and until this theory has been properly worked out, it does not seem advisable to enter upon the discussion of the calculation of the various tables.

The theory of those separable partitions which contain no part equal to unity is, from another point of view, a theory of the covariants of binary forms; in particular, in such cases the general theorem of algebraic reciprocity becomes a remarkable theorem of binary forms which promises to be chiefly of use in the discussion of perpetuants.

The "theorem of symmetric function expressibility" becomes a theorem, of a fundamental character, of covariant expressibility.

It is to be hoped that some of these facts may help to forward the algebraical (as distinct from the symbolical) treatment of the theory of invariants; as yet, however, a purely algebraical demonstration of Gordan's great theorem concerning the finality of the ground covariants seems as far distant as ever. This theorem has been given by Gordan, in a much simplified form, in a tract which appears to be little known; the reference was given to me by Mr. A. R. Forsyth of Trinity College, Cambridge, and it may be useful to give it here: "Ueber das Formensystem binärer Formen." Teubner (Leipzig, 1875).

ROYAL MILITARY ACADEMY, Woolwich, May 9th, 1888.